從**布朗運動**到**華爾街**的**隨機之旅**

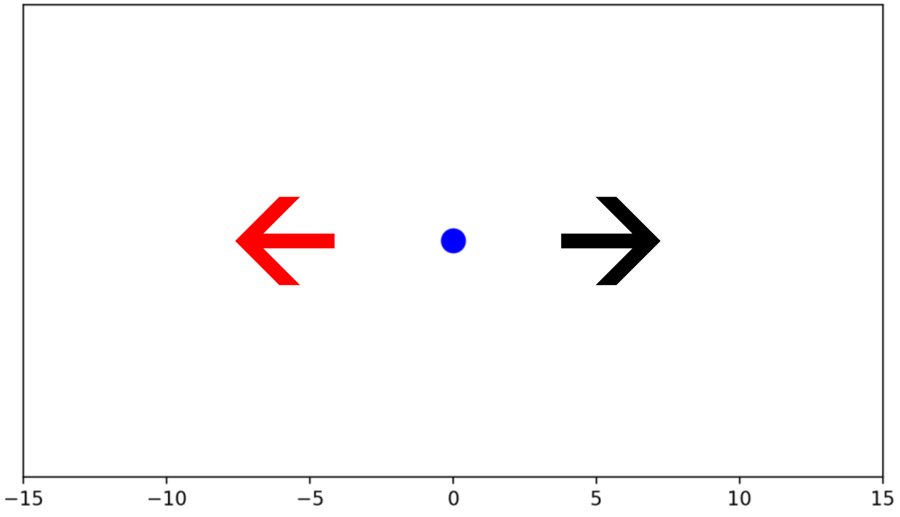
蘊含規律的隨機漫步

IPhOC 2025 **Fun Physics**

小遊戲

- 每個人會拿到一張牌
- 以桌為單位洗牌，念出顏色順序

Current Position



紅色：←

黑色 →

https://media.istockphoto.com/

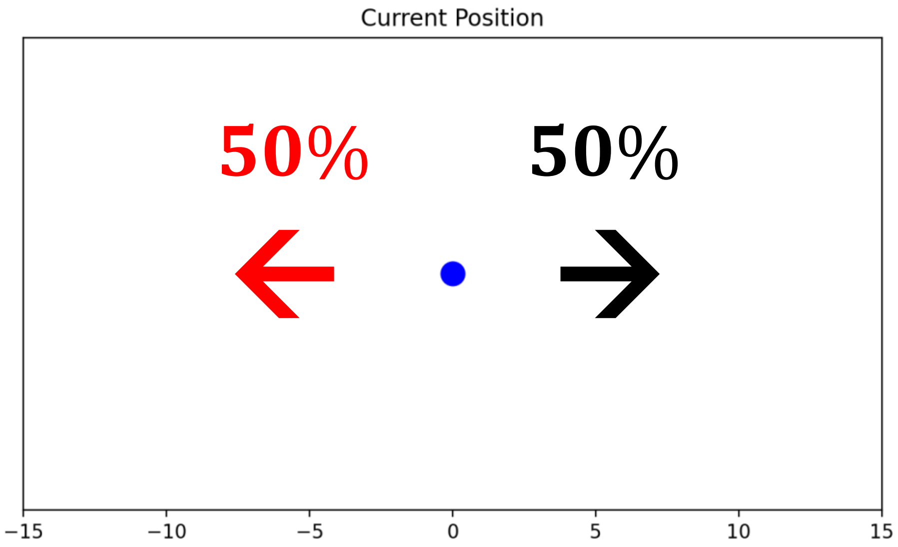
小遊戲 = 隨機漫步 Random Walk

隨機漫步 Random Walk

一維情形：

分子被其他分子碰撞，

每次皆有 50% 機率向左或向右移動一定距離。



歷史

1827

Robert **Brown**

發現花粉在水中的

布朗運動

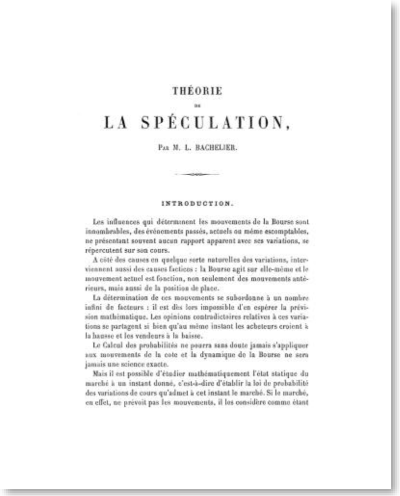
1900

Louis **Bachelier**

第一人使用數學為

股市的隨機性建模

Théorie de la spéculation



THÉORIE
DE
LA SPÉCULATION,
PAR M. L. BACHELIER.

INTRODUCTION.

Les influences qui déterminent les mouvements de la Bourse sont innombrables, des événements passés, actuels ou même escomptables, ne présentant souvent aucun rapport apparent avec ses variations, se répercutent sur son cours.

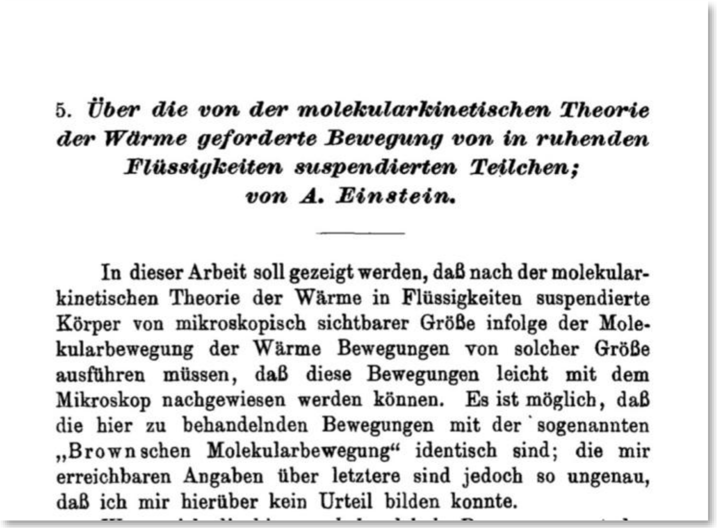
A côté des causes en quelque sorte naturelles des variations, interviennent aussi des causes factices : la Bourse agit sur elle-même et le mouvement actuel est fonction, non seulement des mouvements antérieurs, mais aussi de la position de place.

1905

Albert **Einstein**

首位以**統計物理**探討

布朗運動



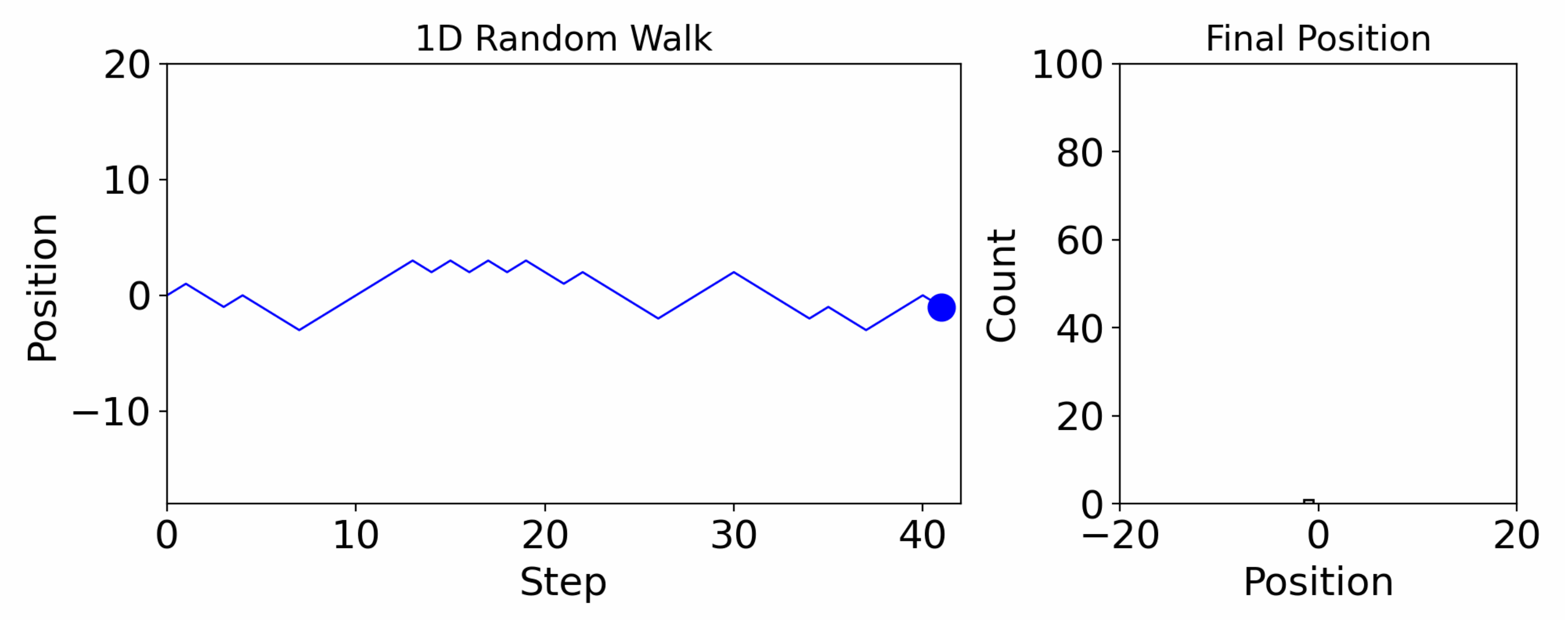
5. *Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen;*
von A. Einstein.

In dieser Arbeit soll gezeigt werden, daß nach der molekularkinetischen Theorie der Wärme in Flüssigkeiten suspendierte Körper von mikroskopisch sichtbarer Größe infolge der Molekularbewegung der Wärme Bewegungen von solcher Größe ausführen müssen, daß diese Bewegungen leicht mit dem Mikroskop nachgewiesen werden können. Es ist möglich, daß die hier zu behandelnden Bewegungen mit der sogenannten „Brownschen Molekularbewegung“ identisch sind; die mir erreichbaren Angaben über letztere sind jedoch so ungenau, daß ich mir hierüber kein Urteil bilden konnte.

https://en.wikipedia.org/wiki/Louis_Bachelier
https://www.google.com/url?sa=i&url=https%3A%2F%2Fx.com%2Fmcnees%2Fstatus%2F1019628375852412931&psig=AOvVaw0YoBBP6wt0lsLj3WSXIeEa&ust=1751118382905000&source=images&cd=vfe&opi=89978449&ved=0CBQQjRxqFwoTCPiZsrvekY4DFQAAAAAdAAAAABAE

模擬

- 走 41 步，每次移動一單位距離
- 重複 500 次，紀錄最終位置



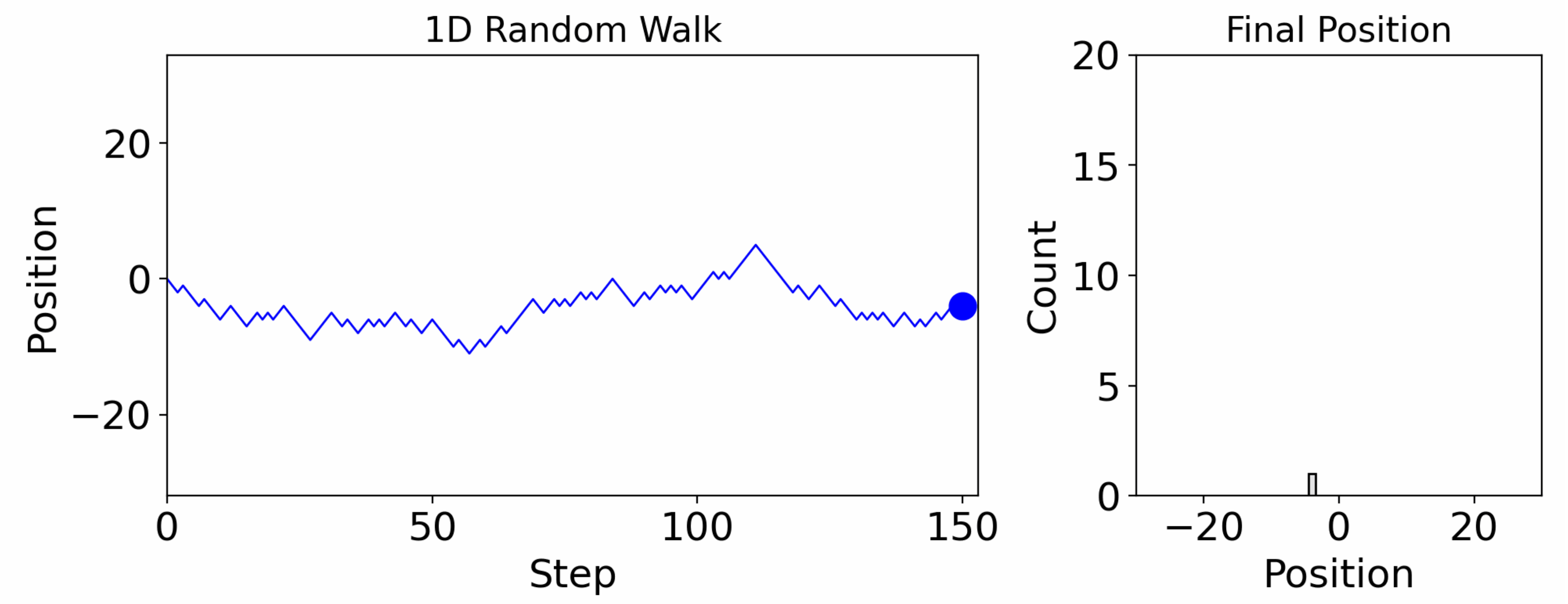
模擬

- 走 t 步，每次移動一單位距離
- 重複 N 次，紀錄最終位置

模擬

- 走 t 步，每次移動一單位距離
- 重複 N 次，紀錄最終位置

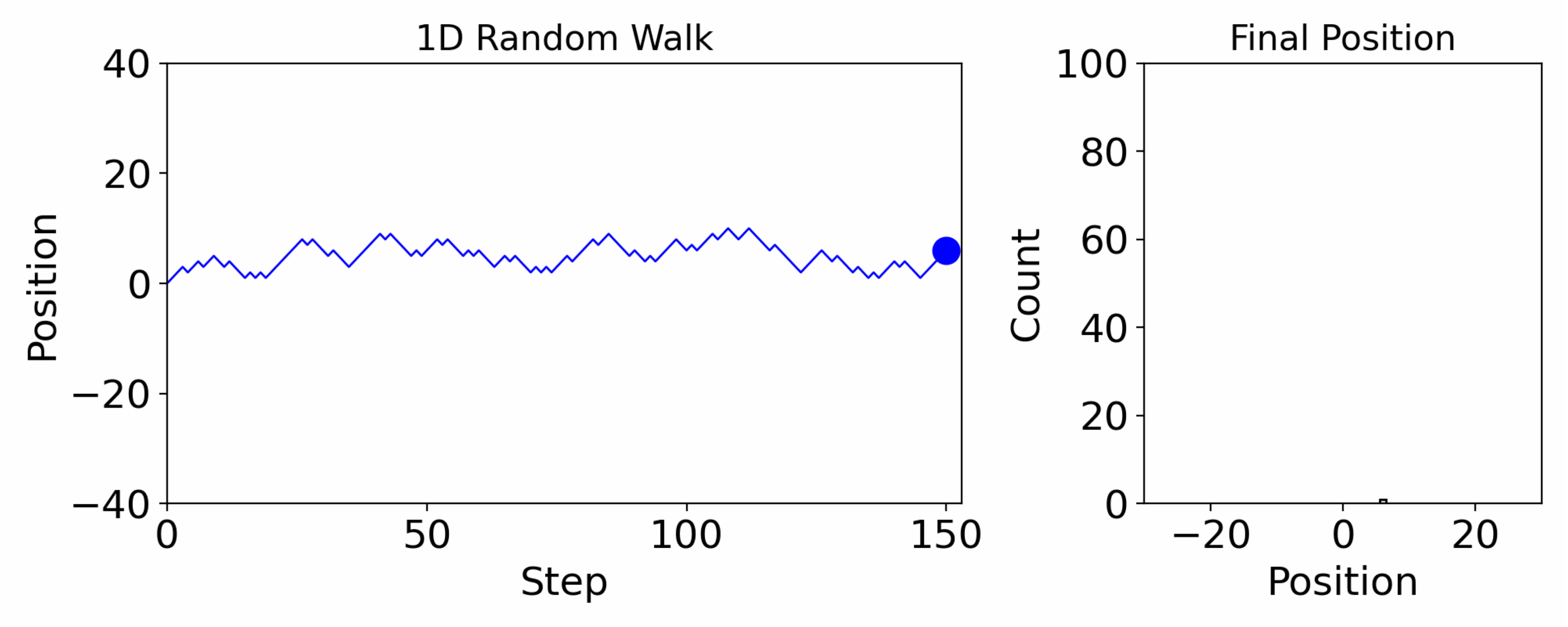
走 t=150 步，重複 N=200 次



模擬

- 走 t 步，每次移動一單位距離
- 重複 N 次，紀錄最終位置

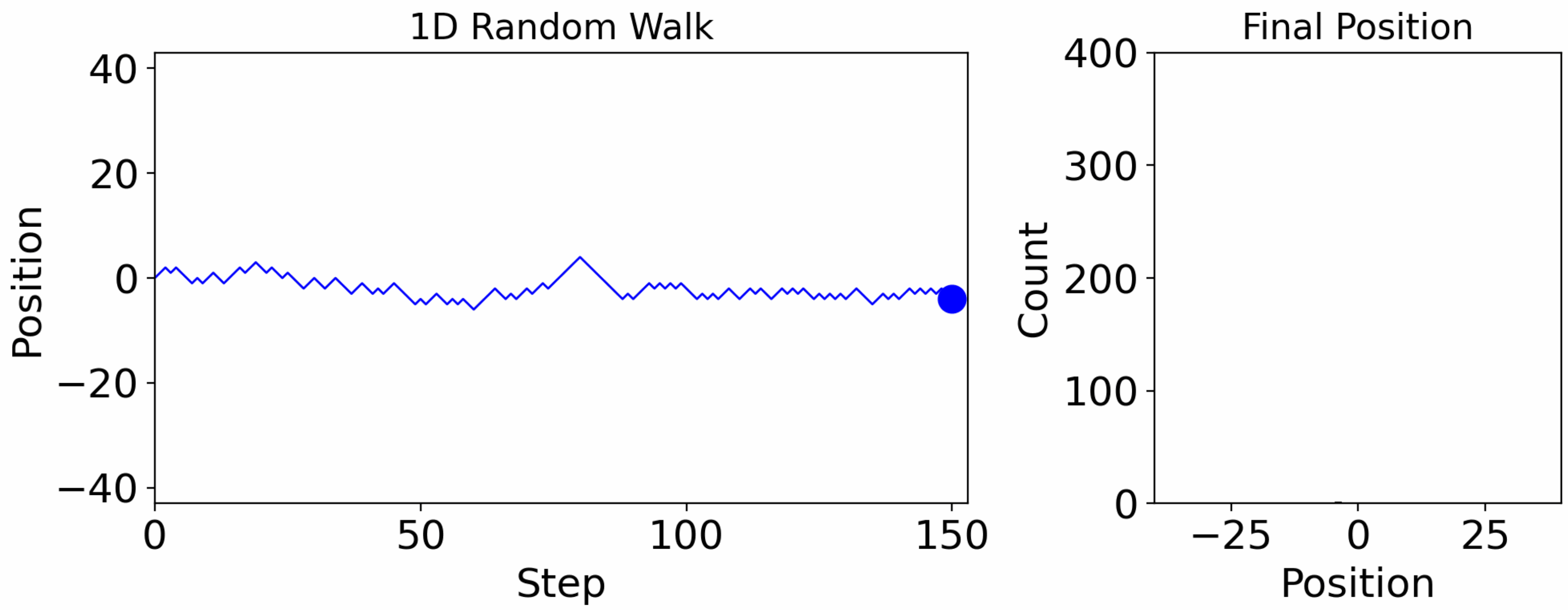
走 t=150 步，重複 N=1000 次



模擬

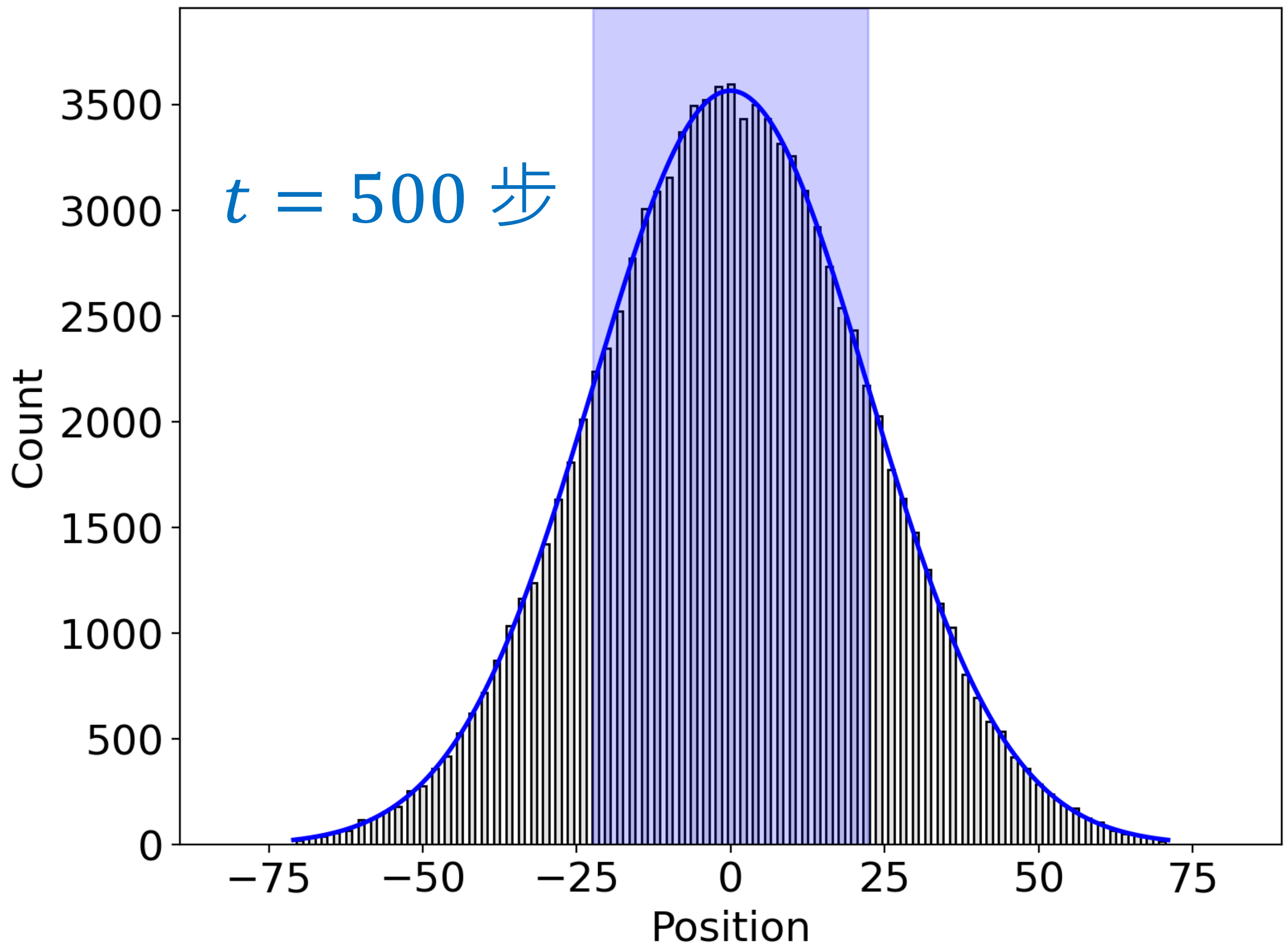
- 走 t 步，每次移動一單位距離
- 重複 N 次，紀錄最終位置

走 t=150 步，重複 N=4000 次



最終位置的分散程度

- 走 **t** 步，每次移動一單位距離
- 重複 $N = 10^5$ 次，紀錄最終位置



最終位置的分散程度

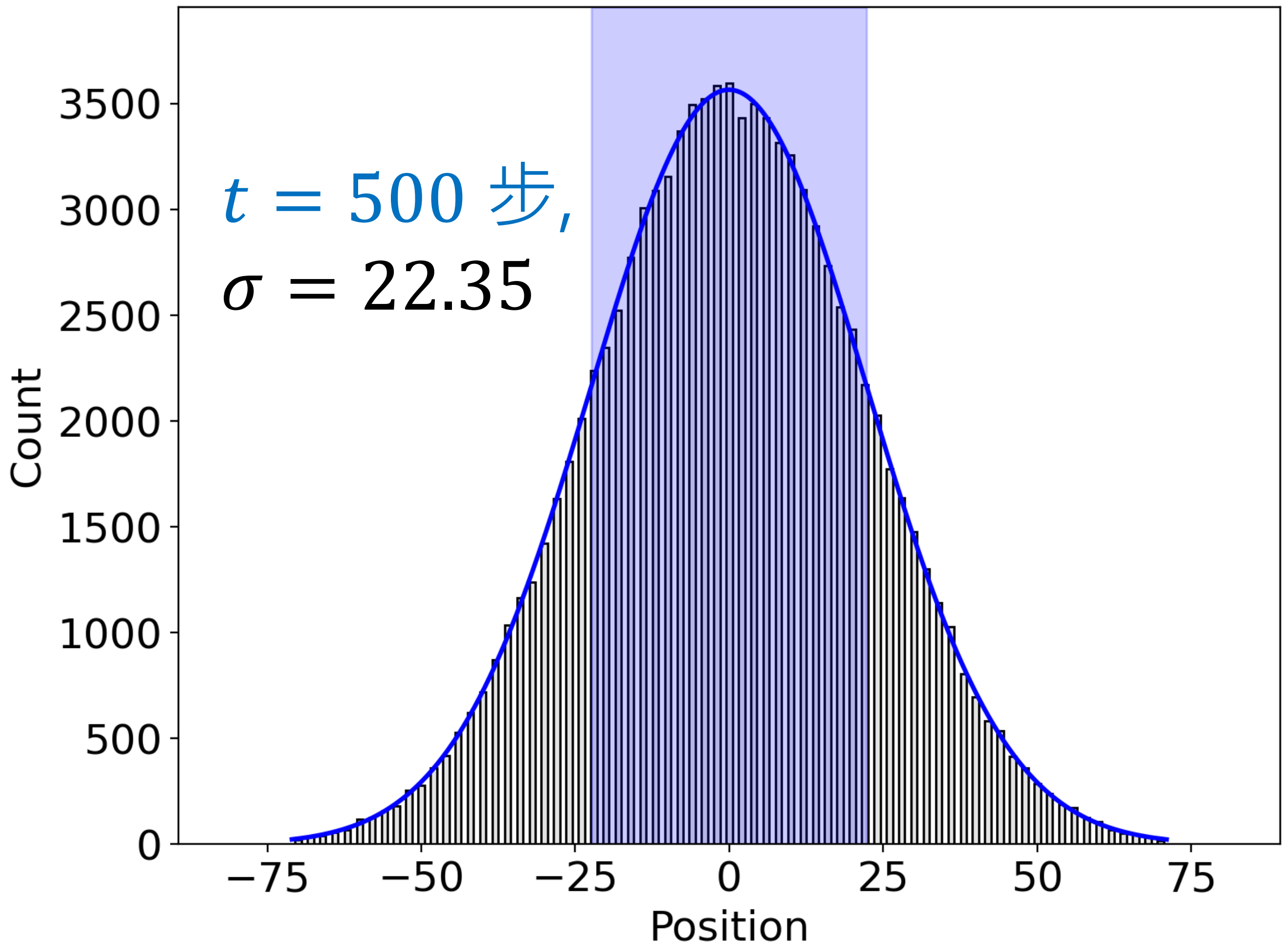
- 走 **t** 步，每次移動一單位距離
- 重複 $\boldsymbol{N = 10^5}$ 次，紀錄最終位置

最終位置：x_i

$$\sigma = \sqrt{\sum \frac{(x_i - \langle x_i \rangle)^2}{n}}$$

$$= \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

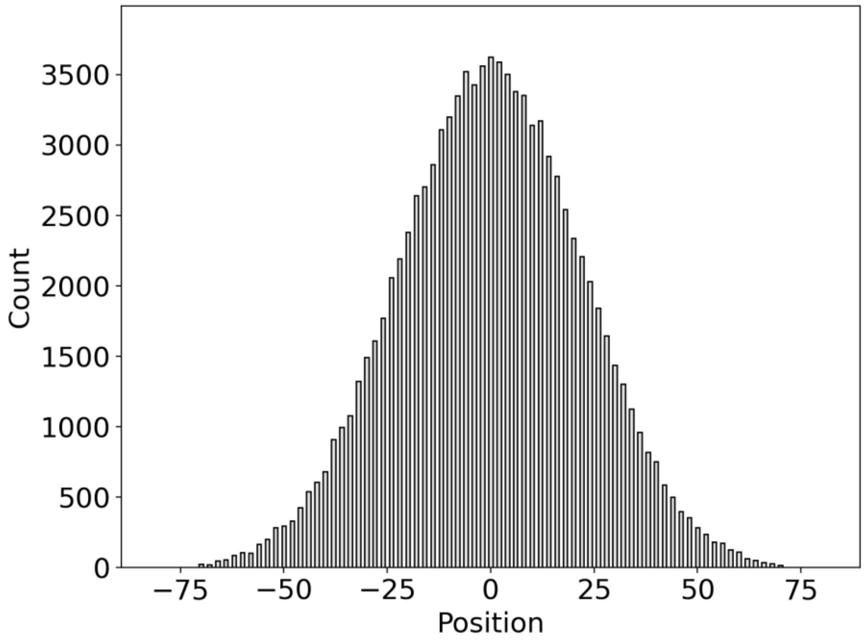
$\boldsymbol{x = -\sigma} \quad \boldsymbol{x = +\sigma}$



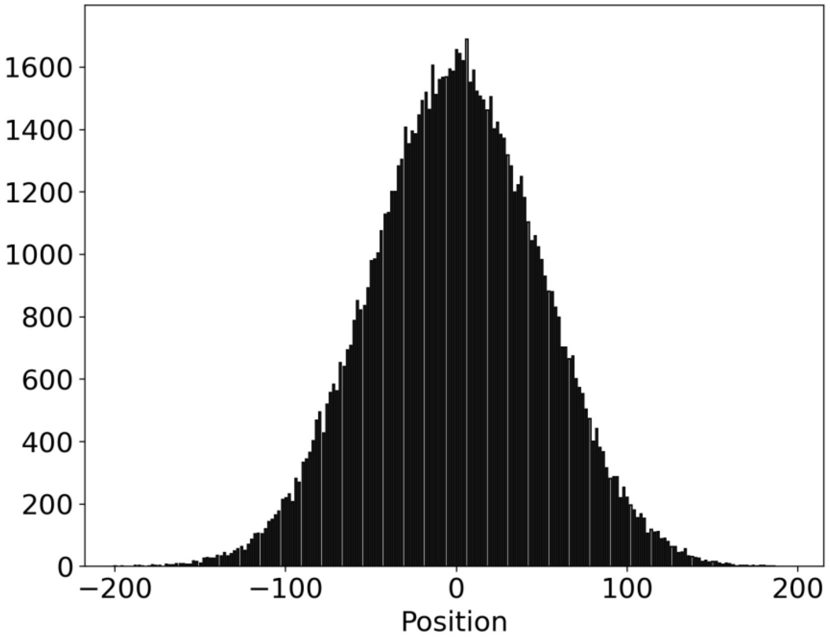
最終位置的分散程度

- 走 t 步，每次移動一單位距離
- 重複 $N = 10^5$ 次，紀錄最終位置

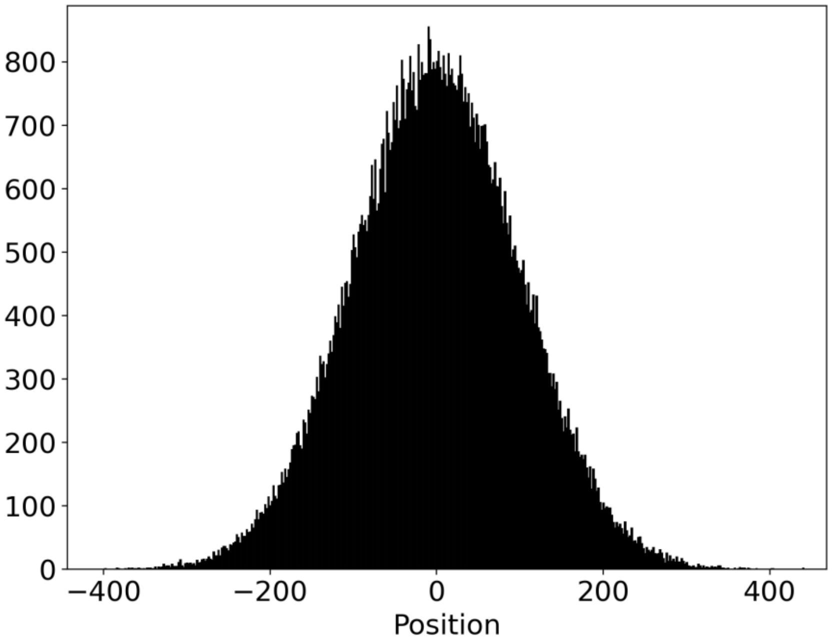
$t = 500$ 步, $\sigma = 22.35$



$t = 2500$ 步, $\sigma = 49.91$



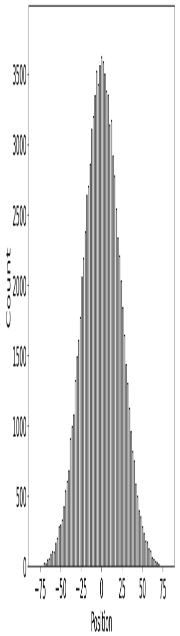
$t = 10^4$ 步, $\sigma = 99.95$

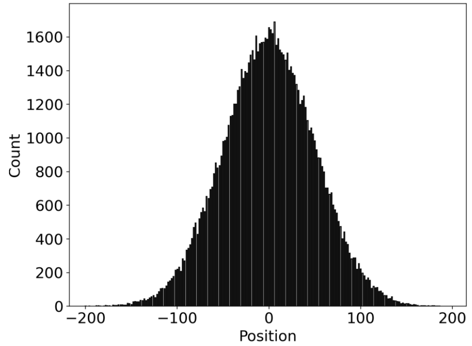


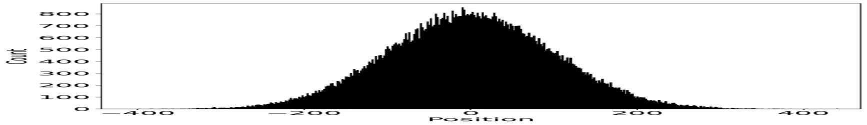
最終位置的分散程度

- 走 t 步，每次移動一單位距離
- 重複 $N = 10^5$ 次，紀錄最終位置

$t = 500$ 步, $\sigma = 22.35$ $t = 2500$ 步, $\sigma = 49.91$ $t = 10^4$ 步, $\sigma = 99.95$







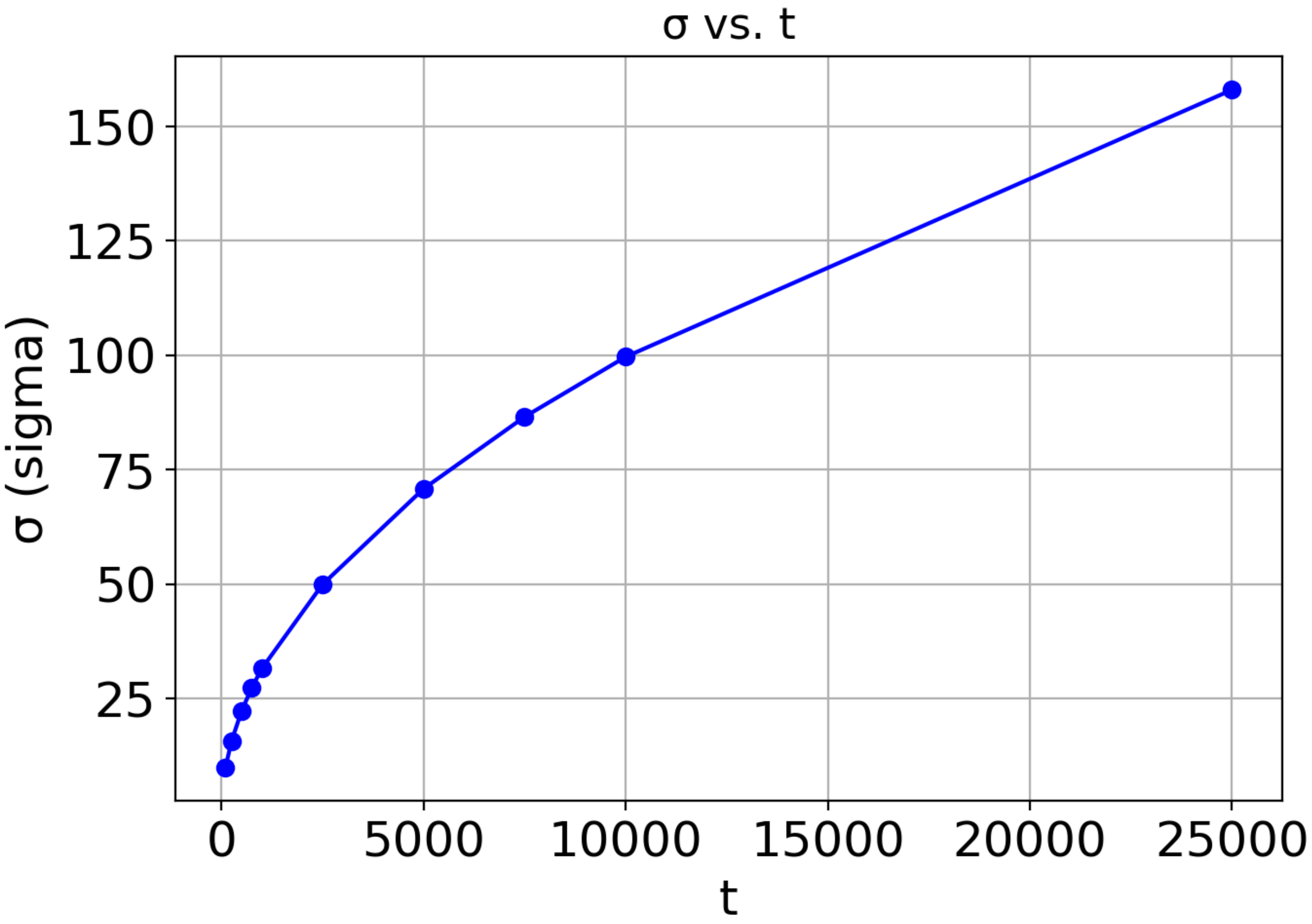
σ, t 正相關, $\sigma \propto t^{\alpha}, \alpha > 0$

決定指數 α

假設 $\sigma(t) = kt^{\alpha} \Longrightarrow \log\sigma(t) = \alpha\log t + \log k$

做 $\log\sigma(t)$ 對 $\log t$ 的散狀圖，
可得斜率為 α 的斜直線

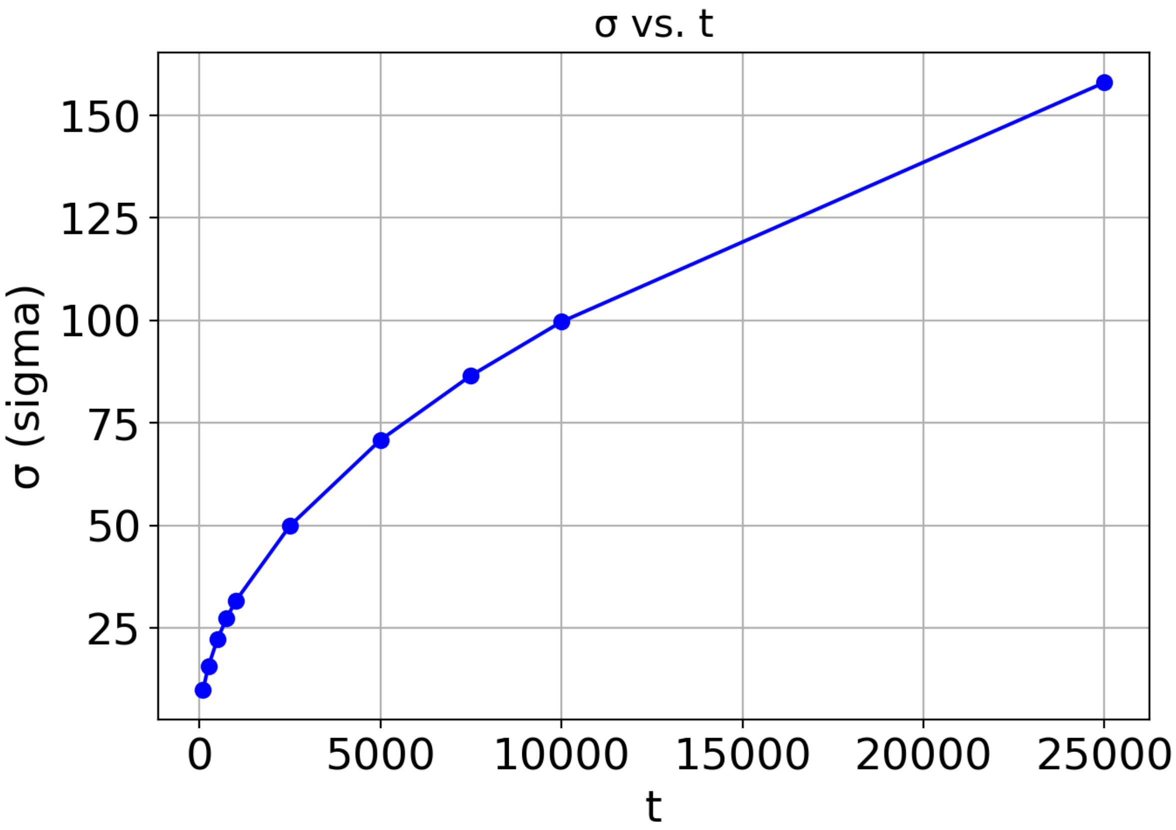
決定指數 α

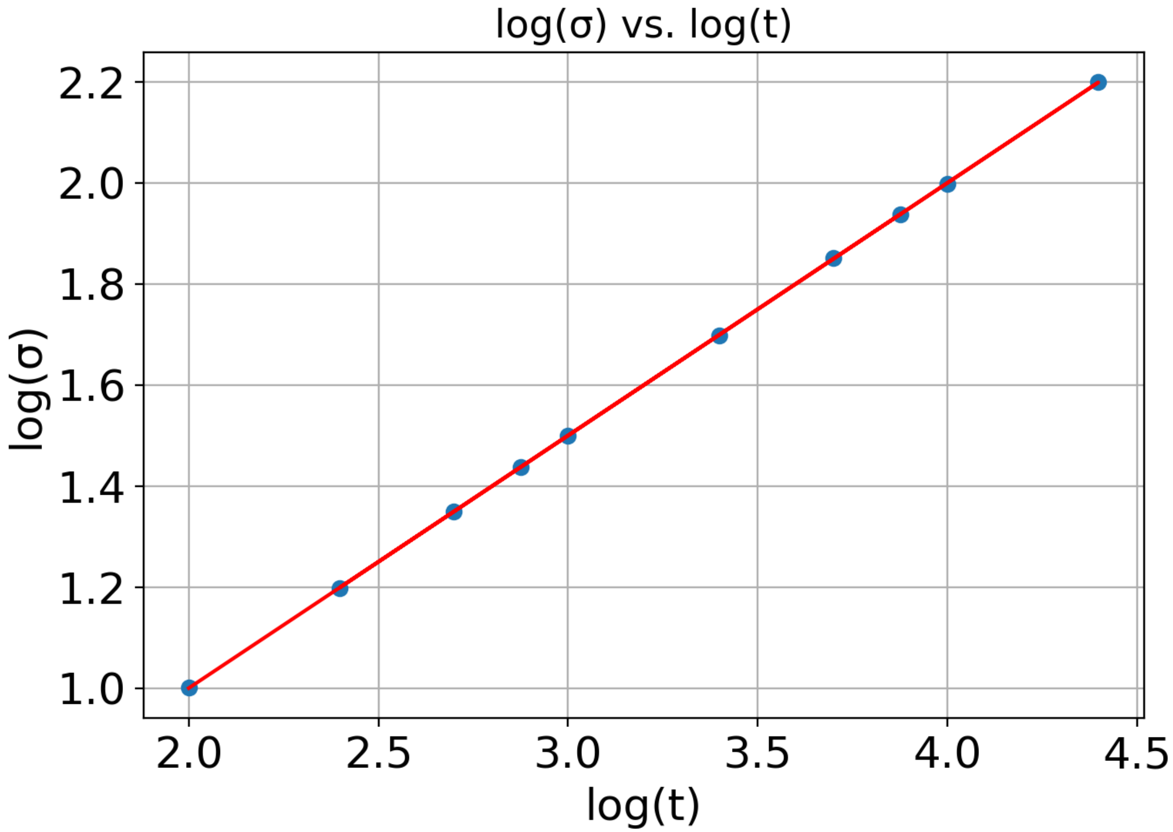

σ vs. t
σ (sigma)
t
150
125
100
75
50
25
0
5000
10000
15000
20000
25000

決定指數 α

$$\log\sigma(t) = \boldsymbol{\alpha}\log t + \log k$$

$$y = (\mathbf{0.49969 \pm 0.00030})x + 0.00088$$



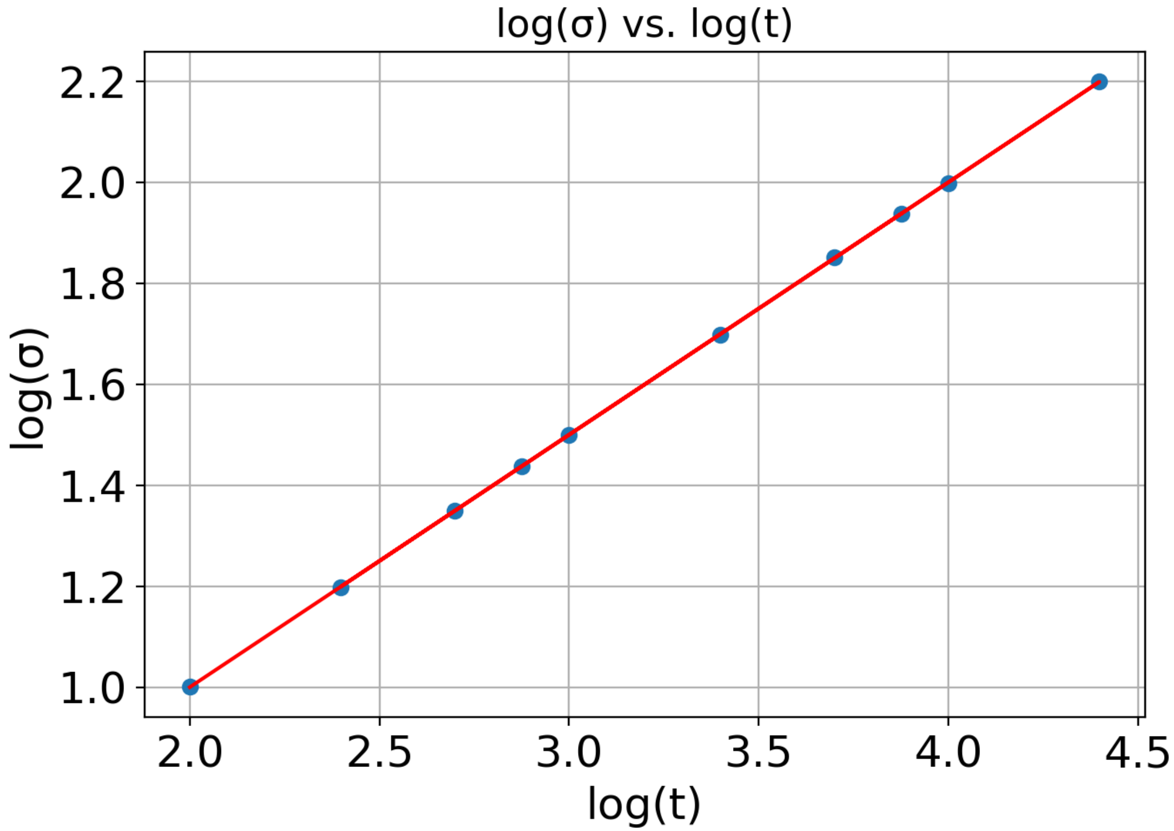


決定指數 α

$$\log \sigma(t) = \boldsymbol{\alpha} \log t + \log k$$

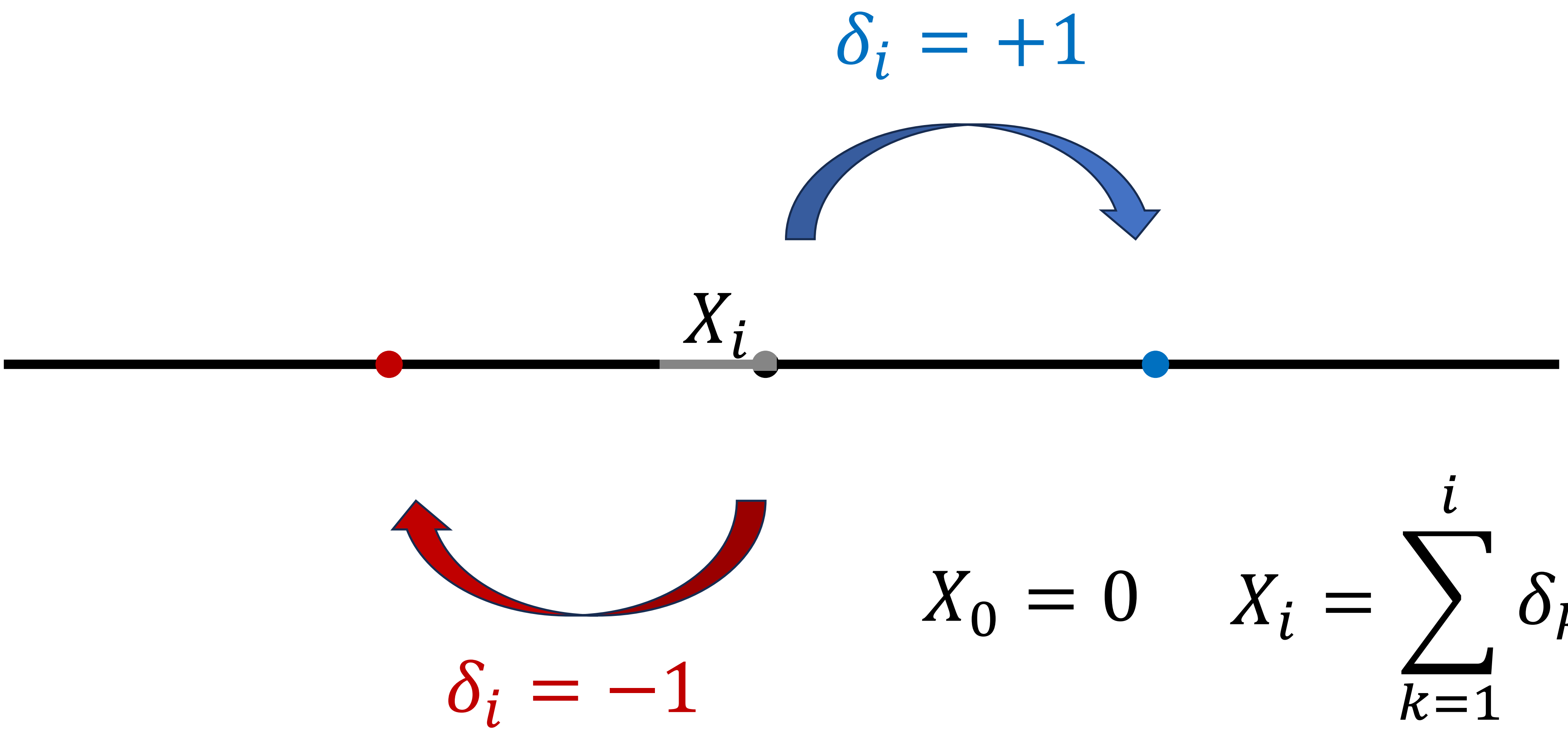
$$\boldsymbol{\alpha = (0.49969 \pm 0.00030) \approx 0.5}$$

$$\boldsymbol{\sigma \propto \sqrt{t}}$$



決定指數 α

$$t = i\tau, \qquad 1 \le i < N, \qquad i \in \mathbb{Z}$$

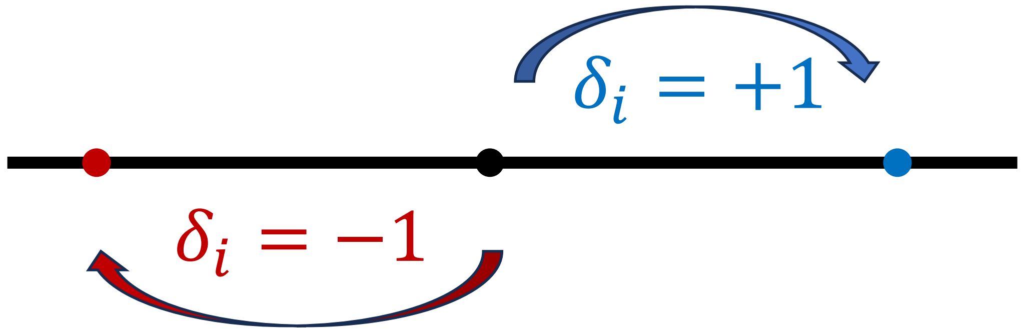


$$X_0 = 0 \qquad X_i = \sum_{k=1}^{i} \delta_k$$

決定指數 α

$$X_i = \sum_{k=1}^{i} \delta_k = \delta_1 + \delta_2 + \delta_3 + \cdots$$

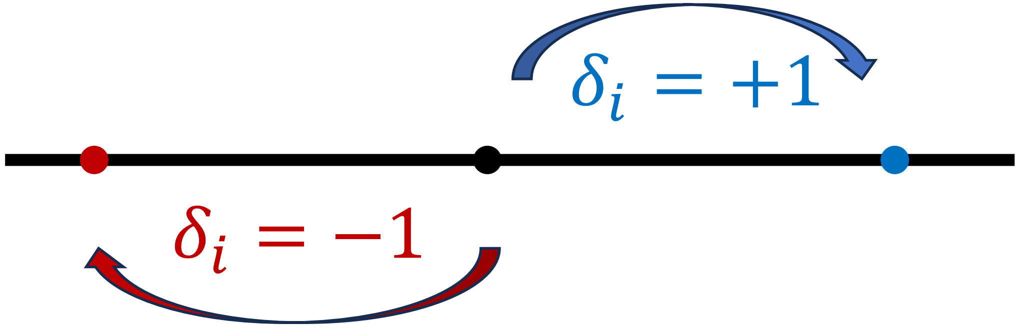
$$\langle X_N \rangle = \langle \delta_1 + \delta_2 + \delta_3 + \cdots \rangle = 0$$



決定指數 α

$$\langle X_N \rangle = 0$$

$$X_i = \sum_{k=1}^{i} \delta_k = \delta_1 + \delta_2 + \delta_3 + \cdots$$



$$\langle X_N^2 \rangle = \langle (\delta_1 + \delta_2 + \delta_3 + \cdots)^2 \rangle$$

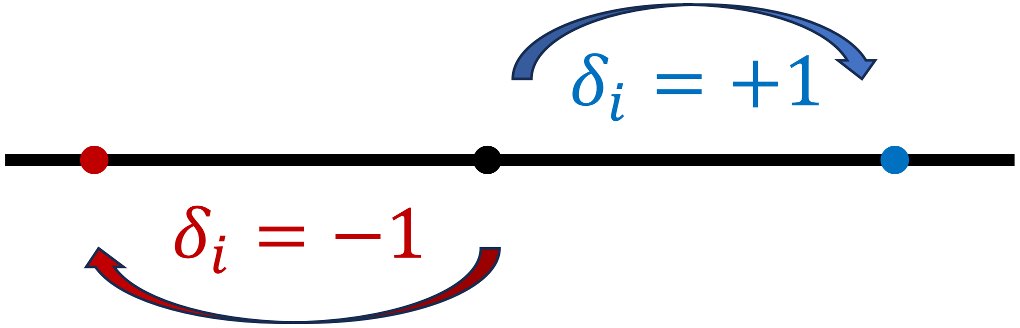
$$= \Big(\underbrace{\langle \delta_1^2 + \delta_2^2 + \cdots \delta_N^2 \rangle}_{=N} + 2\underbrace{\langle \delta_1\delta_2 + \delta_2\delta_3 + \cdots + \delta_{N-1}\delta_N \rangle}_{=0} \Big) = N$$

決定指數 α

$$\langle X_N \rangle = 0$$

$$\langle X_N^2 \rangle = N$$

$$\sigma_{X_N} = \sqrt{\langle X_N^2 \rangle - \langle X_N \rangle^2} = \sqrt{N}$$



過度簡化?

決定比例常數

$$\boldsymbol{\sigma} = \sqrt{\boldsymbol{k}\boldsymbol{t}} \Leftrightarrow \langle \boldsymbol{x}^2 \rangle = \boldsymbol{k}\boldsymbol{t}$$

$$\langle x^2 \rangle = kt$$

決定**比例常數**

Einstein’s Theory

Einstein's Theory

位置
Position

數量密度 $[L^{-1}]$
Number Density

$$\rho(x, t)$$

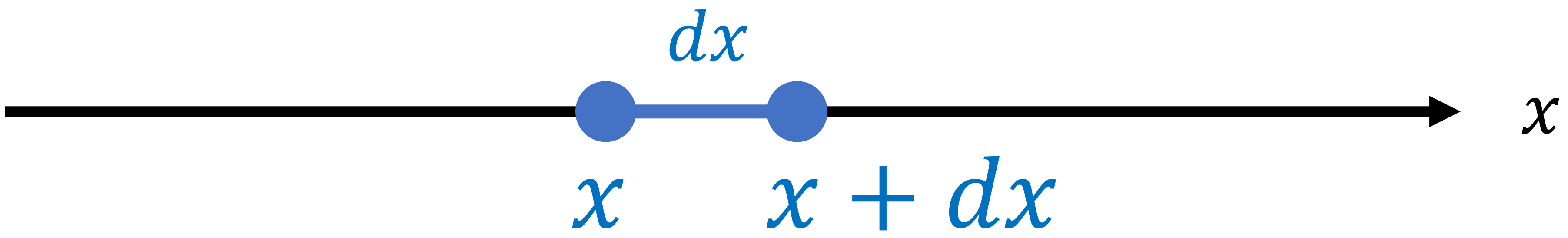
時間
Time

x

Einstein's Theory

$$\rho(x,t)dx$$

時間 t 時,
在位置 x 到 $x+dx$ 間的粒子數



Einstein’s Theory

$$\rho(x, t+\tau)dx$$

時間 $t+\tau$ 時,
在位置 x 到 $x+dx$ 間的粒子數

Einstein's Theory: **泰勒展開**

$$\rho(x, t+\tau) \approx \rho(x, t) + \frac{\partial \rho(x, t)}{\partial t}\tau$$

Einstein's Theory: **泰勒展開**

$$\rho(x, t+\tau) \approx \rho(x,t) + \frac{\partial \rho(x,t)}{\partial t}\tau$$

偏微分

固定 x 時，每單位 t 的改變
會造成多少 $\rho(x,t)$ 的變化?

Einstein’s Theory

位移機率密度 $[L^{-1}]$

Displacement Probability Density

$$\phi(\Delta)$$

位置變化量

Change in Position

Einstein's Theory

$$\phi(\Delta)d\Delta$$

在位置 x 的粒子，於時間 τ 內，位移到 $x+\Delta$ 至 $x+(\Delta+d\Delta)$ 之間的機率

Einstein's Theory: **位移機率密度**

$$\rho(x, t+\tau) = \int_{-\infty}^{\infty} \rho(x-\Delta, t)\phi(\Delta)d\Delta$$

從 $x-\Delta$ 處跳躍 Δ

Einstein's Theory: 泰勒展開

$$\rho(x-\Delta,t) \approx \rho(x,t) + \frac{\partial \rho}{\partial x}(-\Delta) + \frac{1}{2}\frac{\partial^2 \rho}{\partial x^2}(-\Delta)^2$$

Einstein's Theory

$$\rho(x, t+\tau) = \int_{-\infty}^{\infty} \rho(x-\Delta, t)\phi(\Delta)d\Delta$$

$$\rho(x-\Delta, t) \approx \rho(x, t) - \frac{\partial \rho}{\partial x}\Delta + \frac{1}{2}\frac{\partial^2 \rho}{\partial x^2}\Delta^2$$

Einstein's Theory

$$\rho(x, t+\tau) \approx \int_{-\infty}^{\infty} \left(\rho(x,t) - \frac{\partial \rho}{\partial x}\Delta + \frac{1}{2}\frac{\partial^2 \rho}{\partial x^2}\Delta^2 \right) \phi(\Delta) d\Delta$$

Einstein's Theory

$$\rho(x, t+\tau)$$
$$\approx \rho(x,t)\int_{-\infty}^{\infty}\phi(\Delta)d\Delta - \frac{\partial\rho}{\partial x}\int_{-\infty}^{\infty}\Delta\phi(\Delta)d\Delta + \frac{1}{2}\frac{\partial^2\rho}{\partial x^2}\int_{-\infty}^{\infty}\Delta^2\phi(\Delta)d\Delta$$

Einstein's Theory

$$\rho(x, t+\tau)$$
$$\approx \rho(x,t)\int_{-\infty}^{\infty}\phi(\Delta)d\Delta - \frac{\partial\rho}{\partial x}\int_{-\infty}^{\infty}\Delta\phi(\Delta)d\Delta + \frac{1}{2}\frac{\partial^2\rho}{\partial x^2}\int_{-\infty}^{\infty}\Delta^2\phi(\Delta)d\Delta$$

正規化
Normalization

$$\int_{-\infty}^{\infty}\phi(\Delta)d\Delta = 1$$

對稱性
Symmetry

$$\int_{-\infty}^{\infty}\Delta\phi(\Delta)d\Delta = 0$$

Einstein’s Theory

$$\rho(x, t+\tau) \approx \rho(x,t) \times 1 - \frac{\partial \rho}{\partial x} \times 0 + \frac{1}{2}\frac{\partial^2 \rho}{\partial x^2}\int_{-\infty}^{\infty} \Delta^2 \phi(\Delta) d\Delta$$

Einstein's Theory

$$\rho(x, t + \tau) \approx \rho(x, t) + \frac{1}{2}\frac{\partial^2 \rho}{\partial x^2}\int_{-\infty}^{\infty} \Delta^2 \phi(\Delta) d\Delta$$

Einstein's Theory

$$\rho(x, t+\tau) \approx \rho(x,t) + \frac{1}{2}\frac{\partial^2 \rho}{\partial x^2}\int_{-\infty}^{\infty} \Delta^2 \phi(\Delta)\, d\Delta$$

$$\rho(x, t+\tau) \approx \rho(x,t) + \frac{\partial \rho}{\partial t}\tau$$

Einstein’s Theory

$$\frac{\partial \rho}{\partial t}\tau = \frac{1}{2}\frac{\partial^2 \rho}{\partial x^2}\int_{-\infty}^{\infty}\Delta^2\phi(\Delta)d\Delta$$

Einstein's Theory

$$\frac{\partial \rho}{\partial t} = \frac{\partial^2 \rho}{\partial x^2} \int_{-\infty}^{\infty} \frac{\Delta^2}{2\tau} \phi(\Delta) d\Delta$$

Einstein's Theory

擴散方程

Diffusion Equation

$$\frac{\partial \rho(x,t)}{\partial t} = D\,\frac{\partial^2 \rho(x,t)}{\partial x^2}$$

擴散係數

Mass Diffusivity

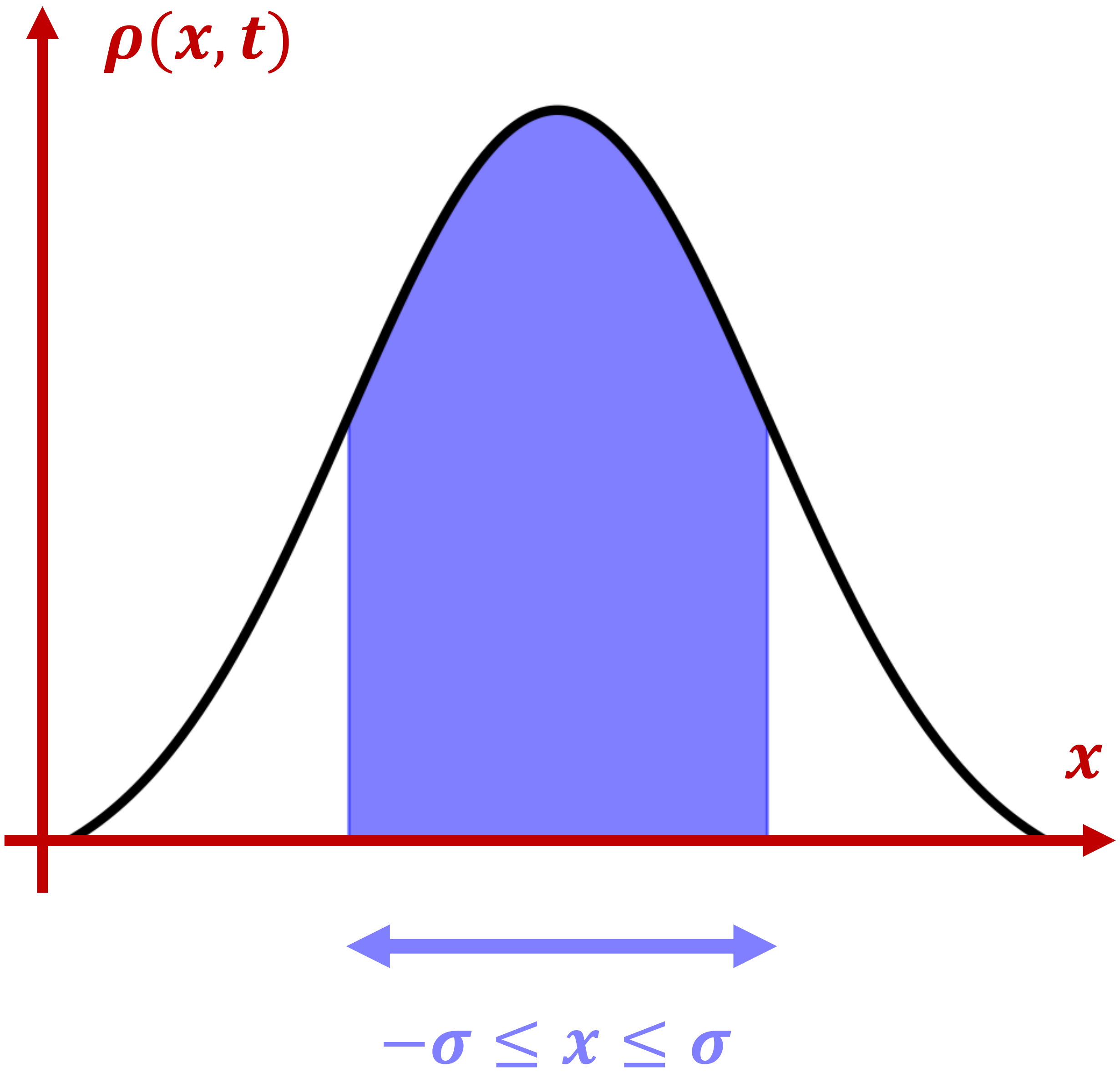
Einstein's Theory

$$\frac{\partial \rho(x,t)}{\partial t} = D\frac{\partial^2 \rho(x,t)}{\partial x^2} \Longrightarrow \rho(x,t) = \frac{N}{\sqrt{4\pi D t}}\exp\left(-\frac{x^2}{4Dt}\right)$$

Einstein's Theory

$$\rho(x,t) = \frac{N}{\sqrt{4\pi D t}} \exp\left(-\frac{x^2}{4Dt}\right)$$

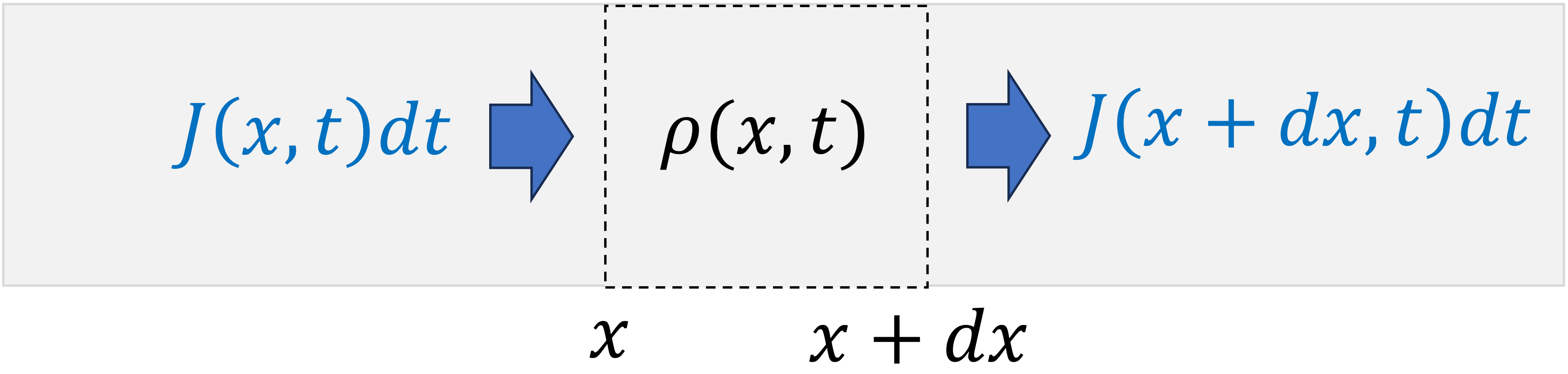
$$\boldsymbol{\sigma = \sqrt{4Dt} \propto \sqrt{t}}$$



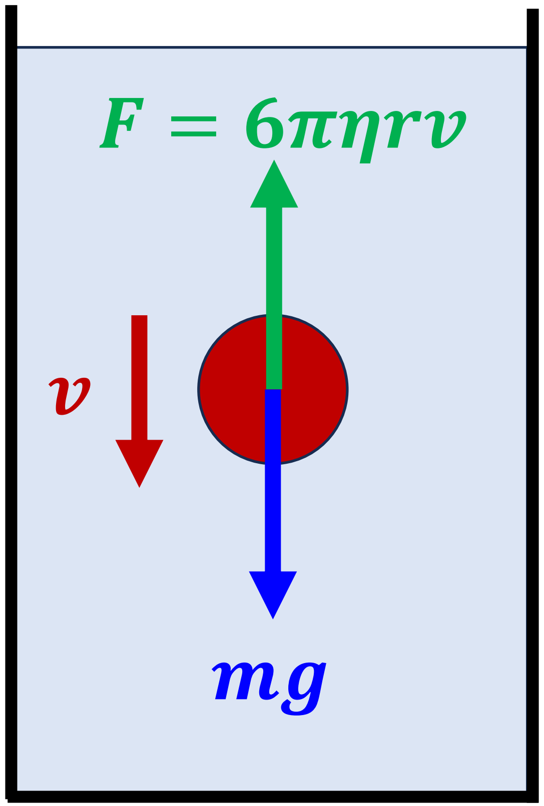
Einstein's Theory: 實驗驗證

$$J(x,t)dt - J(x+dx,t)dt = d\rho(x,t)\cdot dx$$

$$\frac{\partial \rho(x,t)}{\partial t} = D\frac{\partial^2 \rho(x,t)}{\partial x^2} = -\frac{\partial J(x,t)}{\partial x}, \qquad J = -D\frac{\partial \rho(x,t)}{\partial x}$$



Einstein's Theory: 實驗驗證



實驗結果

$$\rho(h) \propto e^{-\frac{mgh}{k_B T}}$$

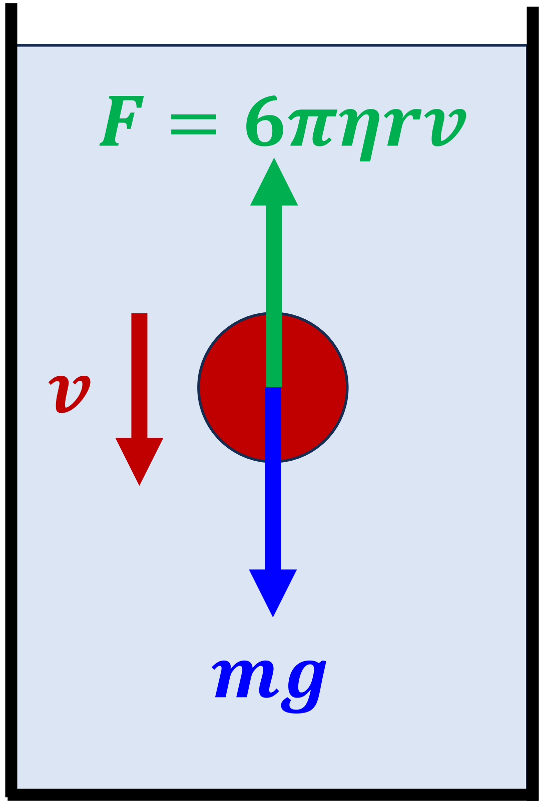
理論驗證

$$D = \mu k_B T = \frac{\mu R T}{N_A} = \frac{\langle x^2 \rangle}{2t}$$

$$J(h) = -D\frac{\partial \rho(h)}{\partial h} \Rightarrow \frac{mgD}{k_B T} = v = \mu m g$$

$$v = \mu m g, \qquad J = \rho v$$

Einstein's Theory: 實驗驗證



Nobel Prize in Physics 1926

Summary

Laureates

Jean Baptiste Perrin

Facts

Biographical

Nobel Prize lecture

Nominations

Photo gallery

Presentation Speech

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Jean Baptiste Perrin
Facts

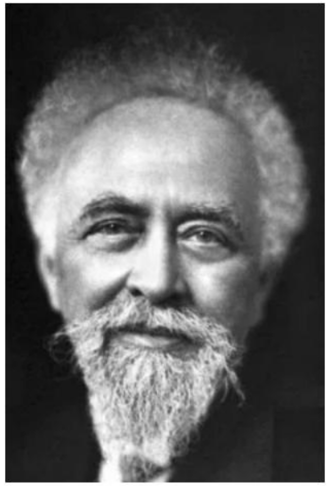


Photo: Henri Martinie. Nobel Foundation archive

Jean Baptiste Perrin
Nobel Prize in Physics 1926

Born: 30 September 1870, Lille, France

Died: 17 April 1942, New York, NY, USA

Affiliation at the time of the award: Sorbonne University, Paris, France

Prize motivation: "for his work on the discontinuous structure of matter, and especially for his discovery of sedimentation equilibrium"

Prize share: 1/1

均值回歸 Mean Reversion

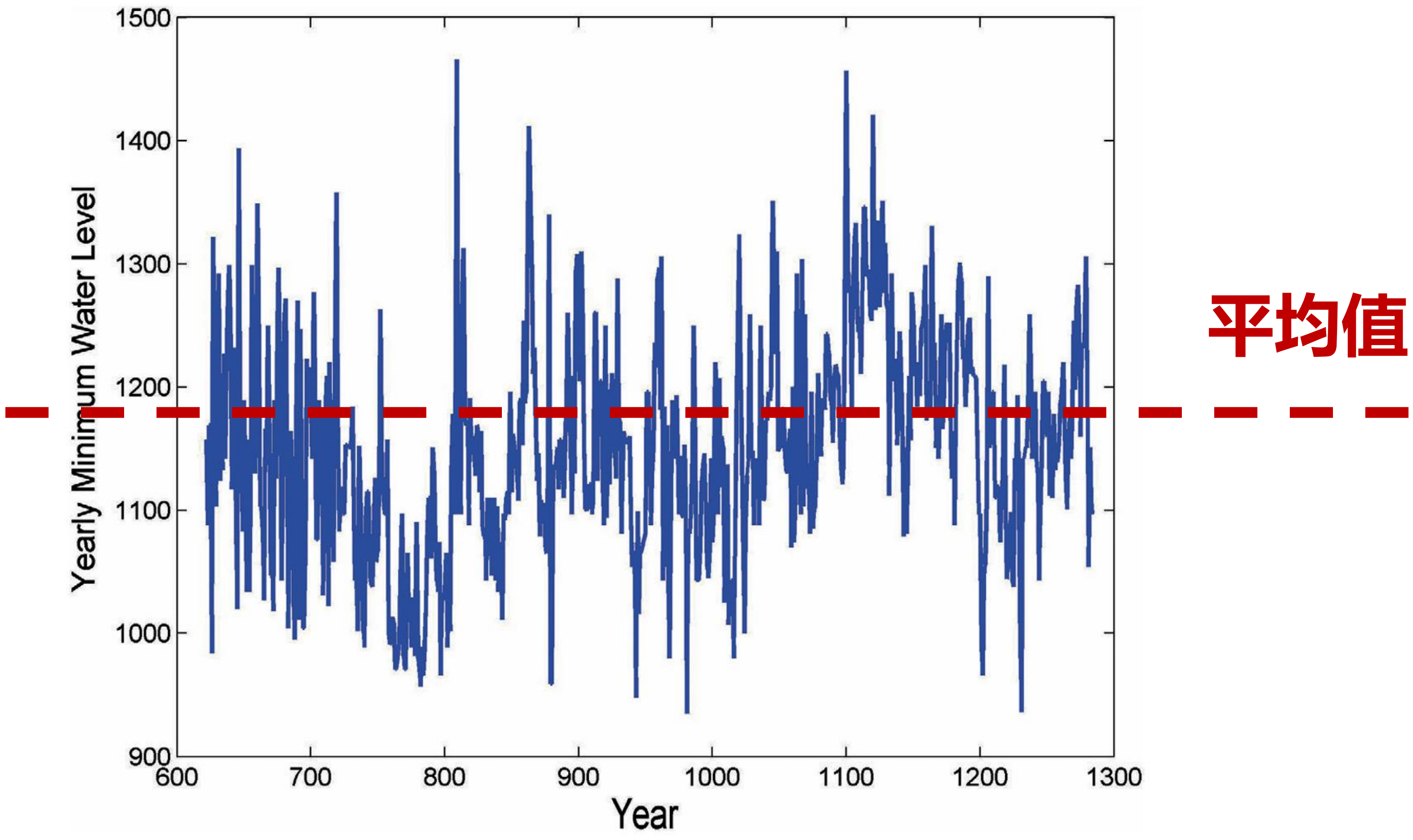
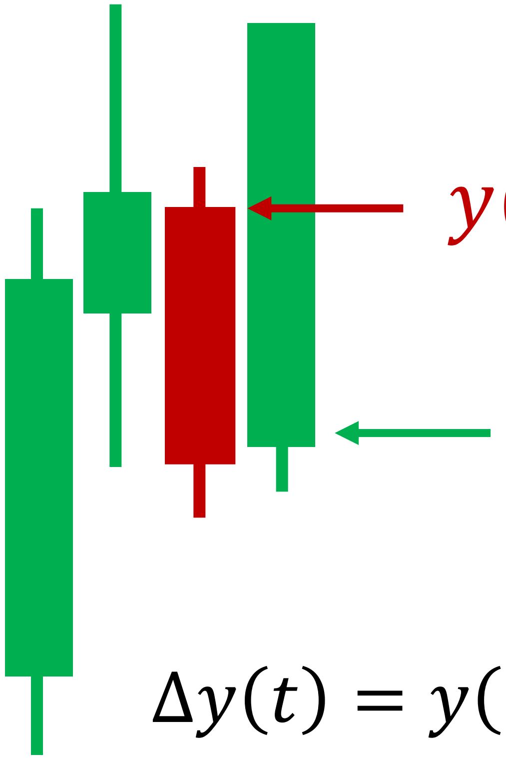


FIGURE 2.1 Minimum Water Levels of the Nile River, 622–1284 AD

均值回歸 Mean Reversion



$y(t-1)$

$y(t)$

$$\Delta y(t) = \lambda\big(y(t-1) - \langle y \rangle\big) + \epsilon$$

$$\lambda < 0$$

$$\Delta y(t) = y(t) - y(t-1)$$

均值回歸 ADF 測試

$$\Delta y(t) = \lambda(y(t-1) - \langle y \rangle) + \epsilon$$

假設 $\lambda = 0$

成立 → **是隨機漫步，沒有均值回歸**

不成立 → **可能有均值回歸**

均值回歸 **方差比值檢驗**

$$\mathbf{Var} = \langle \,|\log(y(t)) - \log(y(t-\tau))\,|^2 \rangle$$

$$\mathbf{Var} \geq \tau?$$

$\xrightarrow{\text{成立}}$ **是隨機漫步，沒有均值回歸**

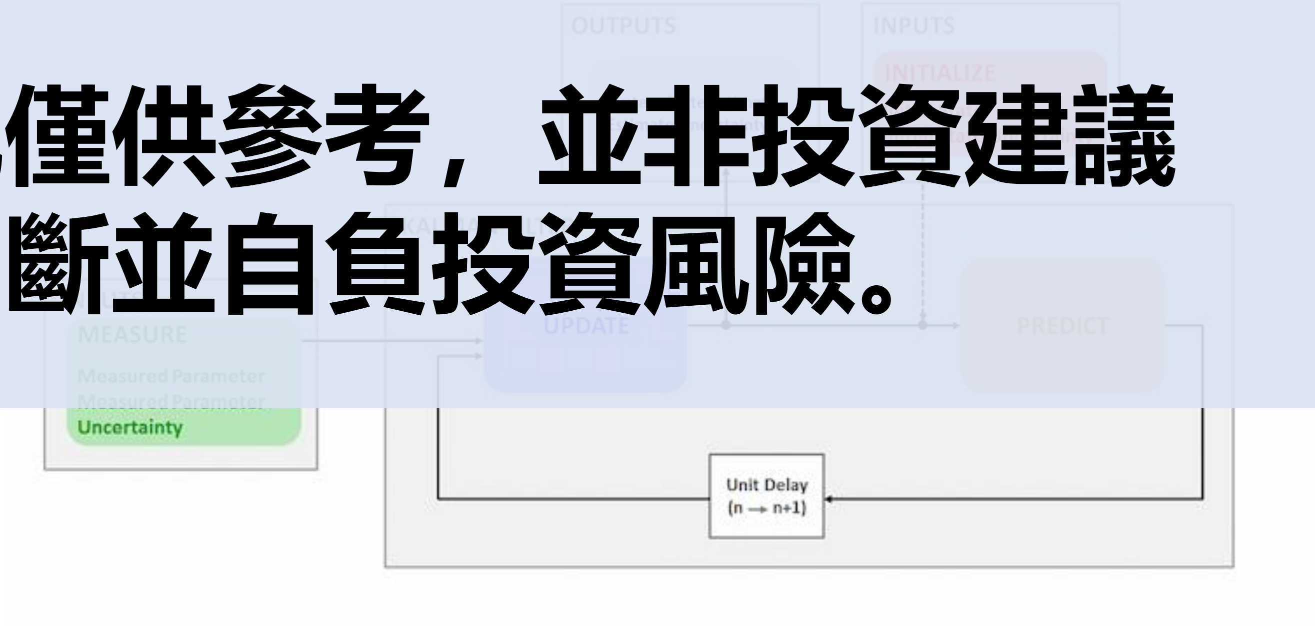
$\xrightarrow{\text{不成立}}$ **可能有均值回歸**

均值回歸的應用

布林通道

https://rich01.com/what-is-bollinger-band/

卡爾曼濾波



https://www.mql5.com/en/articles/17273

本課堂所提供之資訊僅供參考，並非投資建議
投資人應自行判斷並自負投資風險。

擴散方程的解

$$\frac{\partial \rho(x,t)}{\partial t} = D\,\frac{\partial^2 \rho(x,t)}{\partial x^2}$$

$$\rho(x,t) = f(x)\cdot g(t)$$

擴散方程的解

$$\frac{\partial \rho(x,t)}{\partial t} = D\frac{\partial^2 \rho(x,t)}{\partial x^2}$$

$$\rho(x,t) = f(x)\cdot g(t)$$

$$\rho\left(\lambda x, \lambda^2 t\right) = f(\lambda x)\cdot g(\lambda^2 t)$$

$$x \to \lambda x, \qquad t \to \lambda^2 t$$

擴散方程的解

$$\frac{\partial \rho(x,t)}{\partial t} = D\frac{\partial^2 \rho(x,t)}{\partial x^2}$$

$$\rho(x,t) = f(x) \cdot g(t)$$

$$\frac{\partial \rho(x,t)}{\partial t} = f\frac{\partial g}{dt} \qquad \frac{\partial^2 \rho(x,t)}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} g$$

$$\rho\left(\lambda x, \lambda^2 t\right) = f(\lambda x) \cdot g(\lambda^2 t)$$

$$\frac{\partial \rho\left(\lambda x, \lambda^2 t\right)}{\partial t} = \lambda^2 f\frac{\partial g}{dt} \qquad \frac{\partial^2 \rho\left(\lambda x, \lambda^2 t\right)}{\partial x^2} = \lambda^2 \frac{\partial^2 f}{\partial x^2} g$$

擴散方程的解

$$\frac{\partial \rho(x,t)}{\partial t} = f\frac{\partial g}{dt} \qquad \frac{\partial^2 \rho(x,t)}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} g$$

$$\frac{\partial \rho(x,t)}{\partial t} = D\frac{\partial^2 \rho(x,t)}{\partial x^2}$$

$$\frac{\partial \rho\left(\lambda x,\lambda^2 t\right)}{\partial t} = \lambda^2 f\frac{\partial g}{dt} \qquad \frac{\partial^2 \rho\left(\lambda x,\lambda^2 t\right)}{\partial x^2} = \lambda^2\frac{\partial^2 f}{\partial x^2} g$$

擴散方程的解

$$\frac{\partial \rho(x,t)}{\partial t} = D\,\frac{\partial^2 \rho(x,t)}{\partial x^2}$$

$$\rho(x,t) = t^{-\alpha} \cdot h\left(\frac{x}{\sqrt{t}}\right)$$

$$x \to \lambda x, \qquad t \to \lambda^2 t$$

擴散方程的解

$$N=\int_{-\infty}^{\infty}\rho(x,t)dx=\int_{-\infty}^{\infty}\rho\left(\lambda x,\lambda^2 t\right)dx$$

$$=\lambda^{-2\alpha}\lambda\int_{-\infty}^{\infty}\rho(x,t)dx\Rightarrow\alpha=-\frac{1}{2}$$

$$\rho(x,t)=\frac{1}{\sqrt{t}}\cdot h\left(\frac{x}{\sqrt{t}}\right)=\frac{1}{\sqrt{t}}\cdot h(\eta)$$

擴散方程的解

$$\frac{\partial \rho(x,t)}{\partial t} = D\,\frac{\partial^2 \rho(x,t)}{\partial x^2}$$

$$\rho(x,t) = \frac{1}{\sqrt{t}} \cdot h(\eta) = \frac{1}{\sqrt{4Dt}} f(\eta)$$

$$\eta = \frac{x}{\sqrt{4Dt}}$$

擴散方程的解

$$\frac{\partial \rho(x,t)}{\partial t} = D\,\frac{\partial^2 \rho(x,t)}{\partial x^2}$$

$$\rho(x,t) = \frac{1}{\sqrt{4Dt}} f(\eta) \qquad \eta = \frac{x}{\sqrt{4Dt}}$$

擴散方程的解

$$\rho(x,t) = \frac{1}{\sqrt{4Dt}} f(\eta) \qquad \eta = \frac{x}{\sqrt{4Dt}}$$

$$f''(\eta) + 2\eta f'(\eta) + 2f(\eta) = 0$$

擴散方程的解

$$\rho(x,t) = \frac{1}{\sqrt{4Dt}} f(\eta) \qquad \eta = \frac{x}{\sqrt{4Dt}}$$

$$f''(\eta) + 2\eta f'(\eta) + 2f(\eta) = 0 = \frac{d}{d\eta}\left(\frac{df}{d\eta} + 2f\eta\right)$$

擴散方程的解

$$\rho(x,t) = \frac{1}{\sqrt{4Dt}} f(\eta) \qquad \eta = \frac{x}{\sqrt{4Dt}}$$

$$\frac{df}{d\eta} + 2f\eta = \text{const.} = 0$$

擴散方程的解

$$\rho(x,t) = \frac{1}{\sqrt{4Dt}} f(\eta) \qquad \eta = \frac{x}{\sqrt{4Dt}}$$

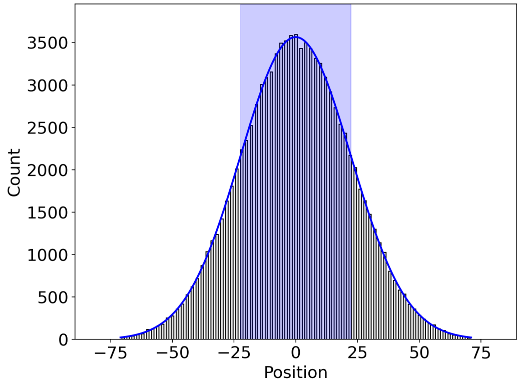
$$\frac{df}{d\eta} = -2f\eta \Longrightarrow \frac{df}{f} = -2\eta d\eta \Longrightarrow f = f_0 e^{-\eta^2}$$

擴散方程的解

$$\eta = \frac{x}{\sqrt{4Dt}}$$

$$\rho(x,t) = \frac{f_0}{\sqrt{4Dt}} e^{-\frac{x^2}{4Dt}} \Longrightarrow \int_{-\infty}^{\infty} \rho(x,t)\,dx = N$$

$$\rho(x,t) = \frac{N}{\sqrt{4Dt}} e^{-\frac{x^2}{4Dt}}$$



其他參考資料

- https://en.wikipedia.org/wiki/Brownian_motion
- Chan, E. (2013). *Algorithmic trading: Winning strategies and their rationale*. John Wiley & Sons.

何承祐 Cheng-You Ho

https://cyh1368.github.io/

上課資源

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